Dynamically Consistent Shallow-Atmosphere Equations with a Complete Coriolis Force

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Shallow-atmosphere equations retaining both the vertical and the horizontal component of the Coriolis force, the latter being neglected in the traditional approximation, are obtained. The derivation invokes Hamilton's principle of least action with an approximate Lagrangian capturing the small increase with height of the solid-body velocity due to planetary rotation. The conservation of energy, angular momentum and Ertel's potential vorticity are ensured in both quasi- and non-hydrostatic systems.

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1. INTRODUCTION

Atmospheric and oceanic motions are usually described and modelled by the traditional primitive equations. These equations are based on two approximations of apparently different nature : the first is the shallow-atmosphere approximation, by which certain metric terms arising in the spherical-coordinate expression of the equations of motion are neglected and the distance r from the origin is replaced by a constant, the planetary radius; the second is the traditional approximation (TA) (Dubois, 1885; Eckart, 1960) and neglects the part of the Coriolis force due to the horizontal component of the planetary rotation rate vector, proportional to cos ϕ at latitude ϕ. Both approximations can be justified separately using order-of-magnitude arguments, depending on the flow under consideration. Even if hydrostatic traditional primitive equations (HPE) are an accurate model to study large-scale atmospheric and oceanic dynamics of the Earth, due to the small aspect ratio of the flow, recent studies have shown that the components of the Coriolis force neglected within the TA may have significant effects under specific circumstances. Gerkema *et al.* (2008) reviewed the general role of the complete Coriolis force for geophysical and astrophysical applications. Several recent studies demonstrate the importance of the non-traditional terms for equatorial circulation both in the ocean and the atmosphere (Hua *et al.*, 1997; Raymond, 2000; Stewart and Dellar, 2011a,b; Hayashi and Itoh, 2012). Therefore a shallow-atmosphere model with a complete representation of Coriolis force could be relevant to study equatorial or other specific flows.

However order-of-magnitude arguments are usually not considered a sufficient rationale for a set of approximate equations of motion : it should also be dynamically consistent in the sense that it possesses conservation principles for mass, energy, absolute angular momentum (AAM) and potential vorticity. For shallow-atmosphere hydrostatic equations, conservation of energy therefore advocates for the neglect of $\cos\phi$ terms (Phillips, 1966). Furthermore a shallow-atmosphere model with a complete Coriolis force lacks a closed absolute angular momentum (AAM) budget (Phillips, 1966). Even though AAM is approximately conserved, i.e. the source terms in the AAM budget are small (Veronis, 1968; Wangsness, 1970), exact conservation is still desirable (Phillips, 1968) in order to be able to analyze the flow (e.g., Eliassen-Palm fluxes) and avoid spurious instabilities that may result from the absence of the dynamical constraints provided by conservation laws. From this point of view the shallow-atmosphere and incomplete Coriolis approximations constitute a dynamically consistent approximation, but only if taken together, not individually.

The TA has been relaxed recently leading to consistent, non-traditional shallow-water or primitive equations with a complete Coriolis force representation but only in a plane geometry (Dellar and Salmon, 2005; Dellar, 2011). Although Dellar (2011) derived the equations from a spherical Hamilton's principle, he then approximated the metric terms so that the geometry appears to be Cartesian. Hence, to the best of our knowledge, no dynamically consistent shallow-atmosphere model with a complete Coriolis force and a spherical geometry is known. Indeed dynamically consistent quasi-hydrostatic equations with a complete Coriolis force can be obtained, but apparently only if the shallow-atmosphere approximation is abandoned and fully spherical geometry is taken into account (White and Bromley, 1995; White *et al.*, 2005). The purpose of this work is to derive such a model, thereby shedding additional light on the nature of the shallow-atmosphere and traditional approximations. In order to guarantee a dynamically consistent model, we follow the approach of Hamilton's principle asymptotics (Holm *et al.*, 2002) : all approximations are performed in the Lagrangian then Hamilton's principle of least action produces the equations of motion following standard variational calculus (Morrison, 1998).

In section 2, the Lagrangian in fully spherical geometry is introduced then approximated according to two small parameters, a shallowness parameter and a planetary Rossby number. In section 3 Hamilton's principle of least action is invoked, leading to a dynamically consistent system of shallow-atmosphere, quasi-hydrostatic or non-hydrostatic equations retaining the non-traditional $\cos\phi$ Coriolis terms. A brief discussion follows in section 4.

2. Approximate Lagrangian for a rapidly-rotating, shallow atmosphere

2.1. Lagrangian for a compressible, rotating flow

The three-dimensional equations of adiabatic fluid motion may be derived using Hamilton's principle of least action. To define the action and express its variations we adopt the Lagrangian point of view : fluid parcels are identified by their Lagrangian label $\mathbf{a} = (a_1, a_2, a_3)$ and their positions are functions of label $\mathbf{a}$ and time τ. A fluid parcel is located by the standard spherical polar coordinates (λ, ϕ, r) respectively, corresponding to unit vectors $(\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{n})$, where $\mathbf{n}$ is the (outward) radial direction. $r\mathbf{n}$ is the position of a fluid parcel and $\dot{\mathbf{r}} = \partial \mathbf{r}(\mathbf{a}, \tau)/\partial\tau = r\dot{\mathbf{n}} + \dot{r}\mathbf{n}$ is its three-dimensional velocity. Furthermore, $\rho_0(\mathbf{r})$ being the initial density field, it is always possible to choose $\mathbf{r}(\mathbf{a}, \tau = 0)$ such that $\rho_0 \det(\partial\mathbf{r}/\partial\mathbf{a}) = 1$ initially. With this choice, the mass of an infinitesimal volume surrounding a fluid parcel is $\mathrm{d}m = \mathrm{d}^3\mathbf{a} = \rho \mathrm{d}^3\mathbf{r}$. Hence at $t > 0$, $\rho = \det\,(\partial\mathbf{r}/\partial\mathbf{a})^{-1}$. Variations $\delta\dot{\mathbf{r}}$ and $\delta\rho$ can be expressed in terms of variations $\delta\mathbf{r}(\mathbf{a}, \tau)$ taken at fixed Lagrangian label (Morrison, 1998).

The Lagrangian for a compressible, rotating flow is the integral over Lagrangian labels :

$$\mathcal{L} = \int \mathrm{d}^3\mathbf{a}\ \left(\mathcal{K}_\mathrm{H} + \delta_\mathrm{NH}\mathcal{K}_\mathrm{V} - \Phi(r) - e(\rho, s)\right) \quad (1)$$

$$\mathcal{K}_\mathrm{H} = \frac{1}{2} r^2\dot{\mathbf{n}}^2 + r^2\dot{\mathbf{n}}\cdot(\mathbf{\Omega}\times\mathbf{n}) \quad (2)$$

$$\mathcal{K}_\mathrm{V} = \frac{1}{2}\dot{r}^2 \quad (3)$$

where $\mathcal{K}_\mathrm{H}$ and $\mathcal{K}_\mathrm{V}$ are the contributions of horizontal and vertical kinetic energy respectively. We have included a switch δ_NH to allow the hydrostatic equations to be considered too. Quasi-hydrostatic equations of motion are obtained by letting $\delta_\mathrm{NH} = 0$, neglecting vertical kinetic energy (White and Bromley, 1995; Roulstone and Brice, 1995). Deep-atmosphere non-hydrostatic equations are obtained if $\delta_\mathrm{NH} = 1$ (White *et al.*, 2005).

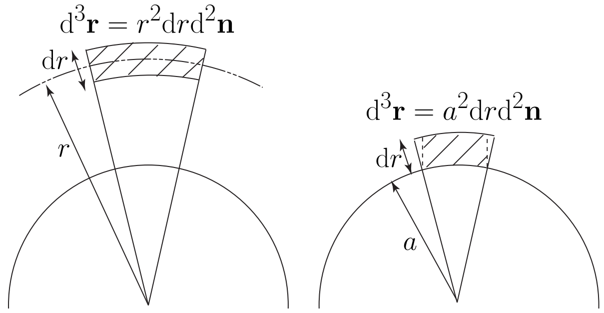

Figure 1. Volume element $\mathrm{d}^3\mathbf{r}$ on the sphere : *deep* (left panel) spherical element vs *shallow* (right panel) spherical element

The term proportional to the constant planetary rotation rate $\mathbf{\Omega}$ gives rise to the Coriolis force. $\Phi(r)$ is the geopotential, assumed spherical here, i.e. we make the spherical geoid approximation and neglect oblate-spheroidal corrections. The internal energy e is a function of density ρ and specific entropy s, and $\partial e/\partial\rho = pv^2, \partial \mathrm{e}/\partial \mathrm{s} = T$ with p, T and $v = 1/\rho$ pressure, temperature and specific volume.

Hamilton's principle of least action states that flows satisfying the equations of motion render the action $\mathcal{S}$ stationary i.e. :

$$\delta\mathcal{S} = \delta\int \mathrm{d}\tau\mathcal{L} = 0 \quad (4)$$

In the next two subsections a simplified Lagrangian approximating (1) in the limit of a shallow atmosphere is introduced, following the approach of Hamilton's principle asymptotics to derive approximate equations of motion (Holm *et al.*, 2002).

2.2. Simplified shallow-atmosphere geometry

The thickness H of the atmosphere and the planetary radius a define a non-dimensional shallowness parameter $\varepsilon = H/a \ll 1$. In this limit a leading-order approximation yields

$$\Phi = gz + O(\varepsilon g H).$$

where we have introduced the elevation $z = r - a = O(H)$ and the constant $\Phi(a)$ has been subtracted from Φ.

Also the spherical volume element is $\mathrm{d}^3\mathbf{r} = r^2\mathrm{d}r\mathrm{d}^2\mathbf{n}$, cf. Fig. 1, left panel, with $\mathrm{d}^2\mathbf{n} = \cos\phi\mathrm{d}\lambda\mathrm{d}\phi$. Correspondingly, the density is obtained as

$$\rho = \det\left(r^2\frac{\partial\left(\mathbf{n}, r\right)}{\partial\mathbf{a}}\right)^{-1}$$

The leading-order approximation for $\varepsilon \ll 1$ is $\mathrm{d}^3\mathbf{r} = a^2\mathrm{d}r\mathrm{d}^2\mathbf{n}$ (*shallow* spherical volume element, cf. Fig. 1, right panel) and

$$\rho = a^{-2}\mathrm{det}\left(\frac{\partial\left(\mathbf{n}, r\right)}{\partial \mathbf{a}}\right)^{-1} \tag{5}$$

(5) implies a corresponding simplification in the expression of the variations $\delta\rho$, yielding $\delta\rho = -\rho(\boldsymbol{\nabla}\cdot\delta\mathbf{n} + \partial_z\delta z)$ where $\boldsymbol{\nabla}$ is the gradient operator along the unit sphere. Therefore integrals involving $\delta\rho$ can be expressed as :

$$\int \mathrm{d}^3\mathbf{a}\, p\frac{\delta\rho}{\rho^2} = \int \mathrm{d}^3\mathbf{a}\,\frac{1}{\rho}\left(\boldsymbol{\nabla}p\cdot\delta\mathbf{n} + \partial_z p\delta z\right) \tag{6}$$

by letting $\delta\rho/\rho^2\,\mathrm{d}m = -\left(\boldsymbol{\nabla}\cdot\delta\mathbf{n} + \partial_z\delta z\right)a^2\mathrm{d}z\mathrm{d}^2\mathbf{n}$, integrating by parts in $z, \mathbf{n}$ and transforming back to an integral with respect to $\mathrm{d}^3\mathbf{a}$ (Dellar and Salmon, 2005).

2.3. Non-traditional horizontal kinetic energy

So far we have introduced approximations that boil down to letting $r = a$ in the expressions of d^3r and $\mathrm{d}\Phi/\mathrm{d}r$, which is the essence of the shallow-atmosphere approximation. The analogous leading-order approximation to the horizontal kinetic energy would be

$$\mathcal{K}_{\mathrm{H}}^{trad} = \frac{1}{2}a^2\dot{\mathbf{n}}^2 + a^2\dot{\mathbf{n}}\cdot(\boldsymbol{\Omega}\times\mathbf{n}) \tag{7}$$

However the smallness of ε is not enough to guarantee that the terms neglected in (7) are small compared to the terms retained in (7). Indeed for a characteristic horizontal velocity U the neglected part of the Coriolis term is $O(\varepsilon a\Omega U)$. If Ω is large enough, this is not necessarily small compared to the retained kinetic energy which is $O(U^2)$. We therefore introduce a planetary Rossby number :

$$\mu = \frac{U}{\Omega a} \ll 1 \tag{8}$$

from which $U^2 = \mu a\Omega U$. Now (7) is a leading-order approximation to (2) only if $\varepsilon \ll \mu$. Otherwise it is safer to retain the next-order Coriolis term, yielding :

$$\mathcal{K}_{\mathrm{H}}^{non\text{-}trad} = \frac{1}{2}a^2\dot{\mathbf{n}}^2 + (a^2 + \underline{2az})\dot{\mathbf{n}}\cdot(\boldsymbol{\Omega}\times\mathbf{n}) \tag{9}$$

where the neglected terms are now of order $\varepsilon U^2 = \mu\varepsilon a\Omega U$ and $\varepsilon^2 a\Omega U$. Assuming $\varepsilon^2 \ll \mu \ll 1$, both neglected terms are now $\ll U^2$ and $\ll \varepsilon a\Omega U$, i.e. smaller than the smallest retained terms.

Notice that assuming $\mu \ll 1$ means only that the typical flow velocity relative to the planet is small compared to the solid-body velocity Ωr due to the planet's rotation, which is generally true for atmospheric and oceanic motions. It does not impose a small Rossby number $U/(\Omega L)$ for flows with a typical scale $L \ll a$. It turns out that letting the Coriolis contribution depend on z (underlined term in (9)) results in a complete Coriolis force (White *et al.*, 2005), while neglecting this dependance yields the traditional Coriolis force. In the next section, all non-traditional contributions are underlined as it is done in expression (9).

3. Non-traditional shallow-atmosphere equations

3.1. Derivation of the equations from Hamilton's principle of least action

Collecting the above approximations to ρ, Φ and $\mathcal{K}_{\mathrm{H}}$ we obtain the approximate Lagrangian :

$$\begin{aligned}\mathcal{L}^{non\text{-}trad} = \int \mathrm{d}^3\mathbf{a}\, \big(&\mathcal{K}_{\mathrm{H}}^{non\text{-}trad}(\mathbf{n}, z, \dot{\mathbf{n}}) \\ &+ \delta_{\mathrm{NH}}\mathcal{K}_{\mathrm{V}}(\dot{z}) - gz - e(\rho, s)\big)\end{aligned} \tag{10}$$

where ρ is defined by the shallow-atmosphere approximation (5) and $\dot{z} = \dot{r}$. Notice that a rigorous asymptotic derivation of $\mathcal{L}^{non\text{-}trad}$ would require estimates of the relative orders of magnitude of all terms retained in, and neglected from (10) (Tort *et al.*, 2013). This can only be done with more information about the flow under consideration, and is not attempted here for the sake of generality.

We can now invoke Hamilton's principle of least action (4) considering the non-traditional Lagrangian $\mathcal{L}^{non\text{-}trad}$. By requiring the stationarity of the action $\mathcal{S}$, we get :

$$\begin{aligned}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{a}\, \Big[&(a^2\dot{\mathbf{n}} + a(a + \underline{2z})\boldsymbol{\Omega}\times\mathbf{n})\cdot\delta\dot{\mathbf{n}} \\ &-a(a + \underline{2z})\boldsymbol{\Omega}\times\dot{\mathbf{n}}\cdot\delta\mathbf{n} + \left(\underline{2a\dot{\mathbf{n}}\cdot(\boldsymbol{\Omega}\times\mathbf{n})} - g\right)\delta z \\ &+\delta_{NH}\dot{z}\delta\dot{z} - \frac{p}{\rho^2}\delta\rho - T\delta s\Big] = 0\end{aligned} \tag{11}$$

where $\delta s = 0$ due to Lagrangian conservation of specific entropy s. Integrating by parts in time and using (6) yields :

$$\begin{aligned}-\int \mathrm{d}\tau\mathrm{d}^3\mathbf{a}\, \Big(&a^2\ddot{\mathbf{n}} + a(a + \underline{2z})\boldsymbol{\Omega}\times\dot{\mathbf{n}} \\ &+\underline{2a\dot{z}\boldsymbol{\Omega}\times\mathbf{n}} + a(a + \underline{2z})\boldsymbol{\Omega}\times\dot{\mathbf{n}} + \frac{1}{\rho}\boldsymbol{\nabla}p\Big)\cdot\delta\mathbf{n} \\ &+\left(\delta_{\mathrm{NH}}\ddot{z} - \underline{2a\dot{\mathbf{n}}\cdot(\boldsymbol{\Omega}\times\mathbf{n})} + g + \frac{1}{\rho}\partial_z p\right)\delta z = 0\end{aligned} \tag{12}$$

Finally factors in front of $\delta\mathbf{n}$ and δz must vanish, yielding the equations of motion. Projecting (12) onto the local basis $(\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{n})$ we get :

$$\begin{aligned}&a^2\left(D_t(\cos\phi\dot{\lambda}) - \sin\phi\dot{\lambda}\dot{\phi}\right) - 2a\Omega\left(a + \underline{2z}\right)\sin\phi\dot{\phi} \\ &\quad+\underline{2a\Omega\dot{z}\cos\phi} + \frac{1}{\rho\cos\phi}\partial_\lambda p = 0, \\ &a^2\left(D_t\dot{\phi} + \sin\phi\cos\phi\dot{\lambda}^2\right) + 2a\Omega\left(a + \underline{2z}\right)\sin\phi\cos\phi\dot{\lambda} \\ &\quad+\frac{1}{\rho}\partial_\phi p = 0, \\ &\delta_{\mathrm{NH}}D_t\dot{z} - \underline{2a\Omega\cos^2\phi\dot{\lambda}} + g + \frac{1}{\rho}\partial_z p = 0,\end{aligned} \tag{13}$$

where the Lagrangian derivative is defined as $D_t = \partial_t + \dot{\lambda}\partial_\lambda + \dot{\phi}\partial_\phi + \dot{z}\partial_z$. The above system (13) constitutes non-traditional shallow-atmosphere equations on the rotating sphere in terms of angular velocities $\left(\dot{\lambda}, \dot{\phi}\right)$. The underlined terms result from the z-dependent

Coriolis contribution present in (9) and absent from (7). They produce non-traditional components of the Coriolis force proportional to $\cos\phi$ (which alter the horizontal and vertical momentum balances), and an $O(\epsilon)$ correction to the traditional Coriolis force proportional to $\sin\phi$ (which alters the horizontal momentum balance only). Non-traditional quasi-hydrostatic equations are obtained with $\delta_{\mathrm{NH}} = 0$.

We can rewrite the system (13) by introducing components of physical velocity $\mathbf{u} = (u, v, w)$: $u = a\cos\phi\dot{\lambda}$, $v = a\dot{\phi}$, $w = \dot{z}$,

$$\begin{aligned} D_t u - \left(2\Omega\left(1 + \underline{\frac{2z}{a}}\right) + \frac{u}{a\cos\phi}\right) v\sin\phi \\ + \underline{2\Omega w\cos\phi} + \frac{1}{\rho a\cos\phi}\partial_\lambda p &= 0, \\ D_t v + \left(2\Omega\left(1 + \underline{\frac{2z}{a}}\right) + \frac{u}{a\cos\phi}\right) u\sin\phi \\ + \frac{1}{\rho a}\partial_\phi p &= 0, \\ \delta_{\mathrm{NH}} D_t w - \underline{2\Omega u\cos\phi} + g + \frac{1}{\rho}\partial_z p &= 0, \end{aligned} \quad (14)$$

where $D_t = \partial_t + \frac{u}{a\cos\phi}\partial_\lambda + \frac{v}{a}\partial_\phi + w\partial_z$. If underlined terms are omitted, either traditional shallow-atmosphere compressible Euler equations ($\delta_{\mathrm{NH}} = 1$) or the HPE ($\delta_{\mathrm{NH}} = 0$) are recovered.

Finally (14) may be cast in vector form :

$$\begin{aligned} D_t\mathbf{u} + 2\mathbf{\Omega}^{non-trad} \times \mathbf{u} + \frac{1}{\rho}\mathrm{grad}p \\ - \delta_{\mathrm{NH}}\mathbf{n}D_t w + g\mathbf{n} &= 0, \end{aligned} \quad (15)$$

where $D_t\mathbf{u}$ and grad take their shallow-atmosphere expression and the Coriolis vector is defined as :

$$\mathbf{\Omega}^{non-trad} = \left(1 + \frac{2z}{a}\right)\Omega\sin\phi\mathbf{n} + \Omega\cos\phi\mathbf{e}_\phi. \quad (16)$$

3.2. Conservation laws

The conservation of total energy, which takes the same form as with deep and shallow systems,

$$\mathcal{E} = \rho\left(\frac{u^2 + v^2 + \delta_{\mathrm{NH}}w^2}{2} + gz + e(\rho, s)\right) \quad (17)$$

results from the invariance of the action with respect to translations in time.It is also easily derived along the lines of White and Bromley (1995) from the vector-form (15).

In order to obtain the expression of potential vorticity and angular momentum, we determine the canonical momenta corresponding to $\mathcal{L}^{non\text{-}trad}$ by considering variations of $\mathbf{u}$:

$$\delta\mathcal{L}^{non\text{-}trad} = \int \mathrm{d}^3\mathbf{a}\,(\mathbf{m}\cdot\delta\mathbf{u}) \quad (18)$$

$$\mathbf{m} = \left(u + \left(1 + 2\underline{\frac{z}{a}}\right)\Omega a\cos\phi\right)\mathbf{e}_\lambda + v\mathbf{e}_\phi + \delta_{\mathrm{NH}}w\mathbf{n}. \quad (19)$$

Then it can be shown, either from the invariance of the action with respect to rotations around the polar axis via the standard Noether theorem, or by a direct computation, that the axial momentum with the density

$$l_z = \rho\mathbf{m}\cdot a\cos\phi\mathbf{e}_\lambda \quad (20)$$

is locally conserved, leading with appropriate boundary conditions to the conservation of $L_z = \int l_z a^2 \mathrm{d}z\mathrm{d}^2\mathbf{n}$. Futhermore, Ertel's potential vorticity is expressed as

$$q = \frac{(\mathrm{curl}\ \mathbf{m})\cdot\mathrm{grad}(s)}{\rho}, \quad (21)$$

where curl and grad take their standard shallow-atmosphere form, i.e. the spherical-coordinate form with $\partial/\partial r = \partial/\partial z$ and $r = a$. That $Dq/Dt = 0$ can be established directly using a general result derived from the particle relabeling symmetry (Salmon, 1988; Morrison, 1998). A direct derivation is also straightforward. First one can check that an equivalent curl-form of (14) is:

$$\partial_t\mathbf{m} + (\mathrm{curl}\ \mathbf{m})\times\mathbf{u} + \mathrm{grad}\ \mathcal{B} = \frac{1}{\rho}\mathrm{grad}p, \quad (22)$$

where $\mathcal{B} = \frac{\mathbf{u}^2}{2} + \Phi$ is the Bernouilli function. Taking the curl of (22), combining with the continuity equation and entropy budget one can obtain $Dq/Dt = 0$ (Vallis, 2006).

4. DISCUSSION

In this note, we have identified a dynamically consistent shallow-atmosphere model which incorporates a complete Coriolis force representation providing a spherical counterpart to similar equations valid in planar geometry (Dellar (2011); see also Staniforth (2012)). The non-hydrostatic and quasi-hydrostatic variants of the presently derived model can now be considered as a fifth and sixth possibility within the family of approximations identified in White *et al.* (2005). The equations have been derived using Hamilton's principle of least action in Lagrangian coordinates leading to a set of equations in their advective form. The system is dynamically consistent in the sense that properly defined energy, angular momentum and Ertel's potential vorticity are conserved without special effort as a consequence of the variational approach.

The three-dimensional approximate Lagrangian $\mathcal{L}^{non\text{-}trad}$ (10) is intermediate between the exact and approximate Lagrangians considered by Dellar (2011) (Eqs. 4.8 and 5.2). Here no approximation is made with respect to the horizontal geometry, in order to retain a spherical domain. $\mathcal{L}^{non\text{-}trad}$ has been approximated within the shallow-atmosphere approximation. Precisely, the underlying assumptions are a small planetary Rossby number μ and a small non-dimensional thickness of the atmosphere ε such as $\varepsilon^2 \ll \mu \ll 1$.

Compared to the traditional equations, non-traditional terms arise in the horizontal and vertical momentum balance. These terms can also be inferred heuristically or by an asymptotic analysis of the deep-atmosphere equations of motion but it is well-known that the resulting set of equations lack a closed angular momentum budget (Phillips, 1966; Veronis, 1968; Phillips, 1968). The key to restore a closed angular momentum budget is to also expand the standard Coriolis force at $O(\varepsilon)$. With this tiny correction, a dynamically consistent model is obtained. This correction restores the non-divergence of the Coriolis vector (16) arising in the vector form (15). In hindsight, this constraint, which follows from the definition of $2\mathbf{\Omega}$ as the curl of planetary velocity, could have been used to infer heuristically expression (16) as discussed in a Cartesian context by Dellar (2011).

From typical values of parameters for the synoptic motions in the Earth's atmosphere and ocean we arrive at the following estimates : $\varepsilon \sim 1.5 \times 10^{-3}$, $\mu \sim 2.1 \times 10^{-2}$ in the atmosphere and $\varepsilon \sim 5 \times 10^{-4}$, $\mu \sim 2.1 \times 10^{-4}$ in the ocean. Thus, our approximations are relevant for studying the dynamics of both Earth's atmosphere and ocean. For the ocean the traditional HPE with omission of the $O(\varepsilon)$ terms, while retaining $O(\mu)$ terms is less justified than for the atmosphere. Therefore, the present model could be relevant to address non-traditional effects arising from the $\cos\phi$ Coriolis part especially concerning ocean dynamics. Indeed a few studies (Hua *et al.*, 1997; Raymond, 2000; Stewart and Dellar, 2011a,b; Hayashi and Itoh, 2012) point to significant dynamical effect of the complete Coriolis force for some particular equatorial flows.

The model derived here is intermediate between the traditional shallow-atmosphere approximation and deep-atmosphere equations, discarding the fully spherical geometry of the latter. This simplification may ease the numerical implementation of the model (compared to a deep-atmosphere model), especially if a general vertical coordinate with a time-dependant elevation of model levels is used. Furthermore the model derived here could be useful to disentangle the effects due to the $\cos\phi$ Coriolis terms from those due to deep spherical geometry, e.g. on the general circulation of the deep planetary atmospheres.

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